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ON THE STABILITY OF THE WEAK PINSKER PROPERTY

BY

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ABSTRACT

We say that a dynamical system has the weak Pinsker property when it possesses decreasing factors of arbitrarily small entropy such that every one of these factors splits off with a Bernoulli complement. We prove here that this property is stable under the taking of factors and of $\bar{d}$ limits.

Introduction

By a dynamical system (X, T) we mean that T is an invertible measure preserving transformation of the Lebesgue space X.

We say that the dynamical system (X, T) satisfies the weak Pinsker property if it is ergodic with finite entropy and if there exist two sequences of partitions of X, H_n and B_n, $n \geqq 1$, such that

$$(H_{n+1})_T \subset (H_n)_T, \tag{1}$$

$$E(H_n, T) \downarrow 0, \tag{2}$$

$$(H_n)_T \perp (B_n)_T, \tag{3}$$

$$(H_n \vee B_n)_T = X, \tag{4}$$

$$\text{the } T^i B_n,\ i \in Z, \text{ are independent.} \tag{5}$$

We remark first, that the $(B_n)_T$ factors can be chosen to be increasing; in fact it follows from proposition 5 in [5] that there exists for every $n \geqq 1$ a partition $\bar{B}_n$ such that $(\bar{B}_n)_T \perp (H_{n+1})_T$, the $T^i\bar{B}_n$, $i \in Z$, are independent and $(\bar{B}_n \vee H_{n+1})_T = (H_n)_T$; and second that $(H_n)_T \downarrow \pi(T)$ the Pinsker algebra of T.

The meaning of this definition is that the "structure" of these systems lies in factors of arbitrarily small entropy and that their randomness is essentially driven by a Bernoulli process. There is now no example known of a system for

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which the weak Pinsker property is not satisfied. But on the other hand, many nontrivial systems are of that type, for instance the Ornstein Shields K automorphisms (as was proved by Ornstein). We prove here that the weak Pinsker property is stable under the taking of factors (Proposition 1), and of $\bar{d}$ limits (Proposition 2). It is not known whether this property is stable under Kakutani equivalence, nor is it known whether when the property is satisfied for T^2 it is satisfied for T.

I can only be inaccurate in describing how helpful Professor Donald Ornstein's generous suggestions and comments have been to me during the preparation of this paper.

The first lemma (the relative Sinai Theorem) is taken from [1, lemma 5] and is stated without proof.

LEMMA 0. *Let I be an abstract partition with a finite number of sets. For every $\varepsilon > 0$ there exists a number $\delta > 0$, which only depends upon ε and the number of sets in I for which the following property is true. Let (X, T) be an ergodic dynamical system and H and B be two finite partitions of X such that*: $d(B, I) < \delta$,

$$|E(B, T \mid (H)_T) - E(I)| < \delta \quad \left((H)_T = \bigvee_{-\infty}^{\infty} T^i H\right), \quad E(T) \geqq E(H, T) + E(I).$$

Then there exists a partition $\bar{B}$ of X such that $|\bar{B} - B| < \varepsilon$, dist $\bar{B}$ = dist I, *the $T^i B$, $i \in Z$, are independent and $(\bar{B})_T \perp (H)_T$.*

LEMMA 1. *Let (X, T) be an ergodic dynamical system and P, H and B be three finite partitions of X together with a positive number ε_1 such that $(H)_T \perp (B)_T$, the $T^i B$, $i \in Z$, are independent and $(H)_T \vee (B)_T \supset^{\varepsilon_1} P$. Let $\varepsilon_2 > 0$. There exists $\delta > 0$ such that for any partition $\tilde{H}$ with the same number of elements as H for which $|\tilde{H} - H| < \delta$ one can find $\tilde{B}$ such that $(\tilde{B})_T \perp (\tilde{H})_T$, the $T^i \tilde{B}$, $i \in Z$, are independent and $(\tilde{H} \vee \tilde{B})_T \supset^{\varepsilon_1 + \varepsilon_2} P$.*

PROOF. Let us first remark that Lemma 1 is immediate from Lemma 0 whenever $E(H \vee B, T) < E(T)$. Let B_1 be a partition of X such that the $T^i B_1$, $i \in Z$, are independent,

$$(H)_T \perp (B_1)_T, \quad (H \vee B_1)_T \supset^{\varepsilon_1 + \varepsilon_2/10} P \quad \text{and} \quad E(H \vee B_1, T) < E(H, T) + E(B, T).$$

(To see that such a partition exists one can proceed as follows: Let N be such that $\bigvee_{-N}^{N} T^i (H \vee B) \supset^{\varepsilon_1 + \varepsilon_2/20} P$. One can modify B to $\bar{B}$ in such a way as to ensure that $\bar{B}$ is $(H \vee B)_T$ measurable, $|\bar{B} - B| < \varepsilon_2/40N$ and $E(\bar{B}) < E(B)$. We thus have that $\bigvee_{-N}^{N} T^i (H \vee \bar{B}) \supset^{\varepsilon_1 + \varepsilon_2/10} P$. Proposition 5 in [5] now tells us that we can find a partition B_1 such that the T^iB_1, $i \in Z$, are independent, $(B_1)_T \perp (H)_T$, and $(B_1 \vee H)_T = (H \vee \bar{B})_T$.) Let $\delta_1 = E(H \vee B, T) - E(H \vee B_1, T)$. Let N_1 be such that $\bigvee_{-N_1}^{N_1} T^i(H \vee B_1) \supset^{\varepsilon_1+\varepsilon_2/5} P$. Take from Lemma 0 the number δ_2 corresponding to $\varepsilon_2/10N_1$. Choose δ_3 such that if $|\tilde{H} - H| < \delta_3$ then $|E(\tilde{H}, T) - E(H, T)| < \min(\delta_2, \delta_1)$. If we select now $\delta = \min(\delta_2, \delta_3, \varepsilon_2/10N_1)$ we get the result by an immediate application of Lemma 0 and of Pinsker's formula. (Notice too that $E(T) \geqq E(H \vee B_1, T)$.)

LEMMA 2. *Let (X, T) be an ergodic dynamical system and P and Q be two finite partitions of X. Then, given a positive integer n and $\delta > 0$, there exists a $(P)_T$-measurable partition $\tilde{Q}$ of X for which*

$$(\alpha) \qquad d\left(\bigvee_1^n T^i(P \vee Q), \bigvee_1^n T^i(P \vee \tilde{Q})\right) < \delta,$$

$$(\beta) \qquad |E(P, T \mid (\tilde{Q})_T) - E(P, T \mid (Q)_T)| < \delta.$$

PROOF. (The idea of this proof is due to Donald Ornstein.)

(1) There exists n_1 and δ_1 such that if $d(\bigvee_1^{n_1} T^i(P \vee Q), \bigvee_1^{n_1} T^i(P \vee \tilde{Q})) < \delta_1$, then $E(P, T \mid (\tilde{Q})_T) < E(P, T \mid (Q)_T) + \delta$ by a simple application of the formula for conditional entropy. So we just need to produce $\tilde{Q}$ satisfying (α) and

$$(\bar{\beta}) \qquad E(P, T \mid (\tilde{Q})_T) > E(P, T \mid (Q)_T) - \delta.$$

Let $a = E(P, T)$, $b = E(Q, T)$, $c = E(P \vee Q, T)$.

(2) The mean ergodic theorem and the Shannon McMillan theorem tell us that there exists an N for which:

(i) One can find a set E_1 of atoms in $\bigvee_1^N T^{-i}(P \vee Q)$, whose union has measure $> 1 - \delta^2/10^6$ such that every atom g in E_1 has measure between $\exp[-N(c \pm \delta/1000)]$ and such that the distribution of n names in the N-name of any g in E_1 is within $\delta/10$ of dist $\bigvee_1^n T^{+i}(P \vee Q)$.

(ii) One can find a set E_2 of atoms in $\bigvee_1^N T^{-i}Q$ whose union has measure $> 1 - \delta/1000$ such that every q in E_2 has measure between $\exp[-N(b \pm \delta/1000)]$ and is covered, except for a set in q of measure smaller than $\delta m(q)/1000$ by $\exp[N(c - b \pm \delta/500)]$ atoms of $\bigvee_1^N T^{-i}(P \vee Q)$ which are in E_1. Call P_q the set of atoms in $\bigvee_1^N T^{-i}P$ which are thus assigned to every q in E_2.

(iii) One can find a set E_3 of atoms in $\bigvee_1^N T^{-i}P$ whose union has measure $> 1 - \delta/1000$ such that every atom p in E_3 has measure between $\exp[-N(a \pm \delta/1000)]$.

(iv) $N > (1000/\delta)\mathrm{Log}(200/\delta)$.

(3) Call $\bar{E}_2$ a maximal set of atoms in $\bigvee_1^N T^{-i}Q$ which lie in E_2 such that it has

been possible to assign to every q in $\bar{E}_2$ a set $\bar{P}_q$ of atoms in $\vee_1^N T^{-i}P$ in such a way as to ensure:

(i) $\bar{P}_q \subset P_q \cap E_3$ for every q in $\bar{E}_2$.

(ii) The number of elements in every $\bar{P}_q$ is greater than $\exp[N(c-b-\delta/100)]$.

(iii) If $q_1 \neq q_2$ are two atoms in $\bar{E}_2$, then $\bar{P}_{q_1} \cap \bar{P}_{q_2} = \emptyset$.

Let E_4 be the union of all the $\bar{P}_q$'s for all q's in $\bar{E}_2$. We claim that $m(E_4) > 1-\delta/100$. Otherwise there would exist in $E_4^c \cap E_3 = J$ a set of measure $\Sigma_{q\in \bar{E}_2} m(q \cap P_q \cap J)$ which is bigger than $\delta/200$ which would be covered by less than $\exp[N(b+\delta/1000)]\exp[N(c-b-\delta/100)] = \exp[N(c-9\delta/1000)]$ atoms of E_1 (the size of any of which is smaller than $\exp[-N(c-\delta/1000)]$ by (2)(i)). This is inconsistent with (2)(iv).

(4) We now pick a set F which is $(P)_T$ measurable as a base of a Rohlin tower of height N such that (i) $m(\bigcup_{i=0}^{N-1} T^iF) > 1-\delta/1000$ and (ii) $m(F \cap E_4) > (1-\delta/50)m(F)$.

We call $G = F \cap E_4$.

(5) Assigning to every p in E_4 the unique q for which $\bar{P}_q$ contains p, we can partition G into less than $\exp[N(a+b-c+\delta/50)]$ sets, each one of them being the union of $p \cap F$ atoms that have been assigned the same q. Using these q names we now build $\tilde{Q}$ which is obviously $(P)_T$ measurable. We have that (3)(i) and (2)(i) imply the conclusion (2) (together with (4)(i) and (4)(ii)). Now if N was chosen large enough so that

$$-\frac{1}{N}\operatorname{Log}\frac{1}{N} - \left(1-\frac{1}{N}\right)\operatorname{Log}\left(1-\frac{1}{N}\right) < \frac{\delta}{100},$$

(5), (4)(i) and (ii) and an elementary computation imply the conclusion $\bar{\beta}$.

LEMMA 3. *Let (X, T) be an ergodic dynamical system and P and Q be two finite partitions of X. Then, given $\varepsilon > 0$, we can find a partition P of X such that $|\tilde{P} - P| < \varepsilon$ and $E(\tilde{P}, T \mid (Q)_T) \leqq (1-\varepsilon/2)E(P, T \mid (Q)_T)$.*

We do not give the proof of this lemma, which is contained in a simple suitable blend of the proofs in [3], proposition 4.4 in [6] and lemma 2 in [2].

LEMMA 4. *Let (X, T) be an ergodic dynamical system and let P, H and B be three finite partitions of X together with a positive number ε such that:*

(1) $(H)_T \perp (B)_T$,

(2) *The T^iB, $i \in Z$, are independent,*

(3) $P \subset^{\varepsilon} (H \vee B)_T$.

Then, for any number $\delta > 0$, there exist two $(P)_T$-measurable partitions of X, $\bar{H}$ and $\bar{B}$ such that:

(4) $$|E(P, T \mid (\bar{H})_T) - E(P, T \mid (H)_T)| < \delta,$$

(5) $$(\bar{H})_T \perp (\bar{B})_T,$$

(6) *The $T^i\bar{B}$, $i \in Z$, are independent,*

(7) $$(\bar{H} \vee \bar{B})_T \supset^{2\varepsilon+\delta} P.$$

PROOF. We first remark that the particular case for which, instead of (3), we assume

(3′) $$P \subset (H \vee B)_T$$

is easier to handle and is a simple refinement of Lemma 2. Lemma 4, such as it stands, is necessary in only one place: the proof of Proposition 2.

(1) The first step in the proof is to produce a finite partition B_1 in X such that

($\bar{1}$) $$(H)_T \perp (B_1)_T,$$

($\bar{2}$) The T^iB_1, $i \in Z$, are independent,

($\bar{3}$) $$(H \vee B_1)_T \supset^{2\varepsilon+\delta/2} P,$$

($\bar{4}$) $$E(P \vee H, T) = E(H, T) + E(B_1, T).$$

(That is, we can add ($\bar{4}$) to the hypotheses (1), (2) and (3).) There exist n_1 and δ_1 such that if $\tilde{P}$ is any partition of X with the same number of elements as P for which

$$d\left(\bigvee_0^{n_1} T^i(\tilde{P} \vee H \vee B), \bigvee_0^{n_1} T^i(P \vee H \vee B)\right) < \delta_1$$

then

(a) $$E(\tilde{P} \vee H, T) < E(P \vee H, T) + \frac{\delta}{500}\left(\frac{E(P \vee H, T) - E(H, T)}{1000}\right),$$

(b) $$|\tilde{P} - P| < 2\varepsilon + \frac{\delta}{500}.$$

The mean ergodic theorem implies that there exists an integer n_2 such that $\delta_1^2/10^6$ almost every atom in $\bigvee_0^{n_2} T^{-i}(P \vee H \vee B)$ has a distribution of n_1 names which is within $\delta_1/500$ of dist $\bigvee_0^{n_1} T^i(P \vee H \vee B)$. Therefore we have that $\delta_1/1000$ almost every g in $\bigvee_0^{+n_2} T^{-i}(H \vee B)$ contains a good atom in $\bigvee_0^{+n_2} T^{-i}(P \vee H \vee B)$.

Using this as an assignment, if we pick a $(B \vee H)_T$ measurable Rohlin tower of height n_2 which fills more than $1-\delta_1/1000$ of the space, such that its base intersects the good set in $\bigvee_0^{n_2} T^{-i}(H \vee B)$ on more than $1-\delta_1/500$ of its measure, we can construct a partition $\tilde{P}$ of X which is $(H \vee B)_T$-measurable for which (a) and (b) are true. Then, Lemma 3 and (a) imply that we can find a partition $\bar{P}$ of X, $(H \vee B)_T$-measurable, such that

(c) $$|\bar{P}-\tilde{P}|<\frac{\delta}{500}$$

and

(d) $$E(\bar{P} \vee H, T) \leqq E(P \vee H, T).$$

Proposition 5 in [5] implies that we can find a $(B \vee H)_T$-measurable partition $\bar{B}$ such that $(\bar{B})_T \perp (H)_T$, $(\bar{B} \vee H)_T = (\bar{P} \vee H)_T$, and the $T^i\bar{B}$, $i \in Z$, are independent. Lemma 0, (b), (c) and (d) then imply that we can produce B_1 satisfying $(\bar{1})$, $(\bar{2})$, $(\bar{3})$ and $(\bar{4})$.

(2) We are now going to prove the lemma assuming that

(4) $$E(P \vee H, T) = E(H, T) + E(B, T).$$

Working the same way as in the proof of Lemma 1, we can produce $\bar{B}_1$, partition of X, in order to ensure

(α) $P \subset^{2\varepsilon+\delta/1000} \bigvee_{-n/2}^{n/2} T^i(H \vee \bar{B}_1)$,

(β) The $T^i\bar{B}_1$, $i \in Z$, are independent,

(γ) $(H)_T \perp (\bar{B}_1)_T$,

(δ) $E(\bar{B}_1) < E(B)$ (call $u = E(B) - E(\bar{B}_1)$).

(3) We know from Lemma 0 that there exists $\bar{\delta}_1$ such that if $\bar{H}$ and $\tilde{B}$ are two $(P)_T$-measurable partitions for which $d(\tilde{B}, \bar{B}_1) < \delta_1$,

$$E(P, T) - E(\bar{H}, T) > E(\bar{B}_1), \quad |E(\tilde{B}, T \mid (\bar{H})_T) - E(\bar{B}_1)| < \bar{\delta}_1,$$

then we can find a $(P)_T$-measurable partition $\bar{B}$ such that (a) $|\bar{B}-\tilde{B}| < \delta/1000n$, (b) dist $\bar{B}$ = dist $\bar{B}_1$, (c) the $T^i\bar{B}$, $i \in Z$, are independent, and (d) $(\bar{B})_T \perp (\bar{H})_T$. Let us call $\bar{\delta} = \inf(\delta/1000, u/1000, \bar{\delta}_1/1000)$, $a = E(P, T)$, $b = E(H, T)$ and $c = E(P \vee H, T)$.

(4) We see now, by the same procedure as in the proof of Lemma 2, that we can find an integer N such that:

(a) There is a set E_1 of atoms in $\bigvee_0^{N-1} T^{-i}(P \vee H \vee \bar{B}_1)$ the measure of which is greater than $1-\bar{\delta}^2/4\cdot 10^6$ such that every atom in E_1 has a distribution of n-names which is within $\bar{\delta}/1000$ of dist $\vee_1^n T^i(P \vee H \vee \bar{B}_1)$.

(b) There is a set $\bar{E}_1$ of atoms of $\vee_0^{N-1} T^{-i}(P \vee H)$ whose union has measure $>1-\bar{\delta}^2/4\cdot 10^6$ such that for every g in $\bar{E}_1$, $m(g)$ is between $\exp[-N(c \pm \bar{\delta}/1000)]$.

(c) There is a set E_2 of atoms in $\vee_0^{N-1} T^{-i}\bar{B}_1$ the measure of which is $>1-\bar{\delta}^2/4\cdot 10^6$ such that for any b in E_2, $m(b)$ is between $\exp[-N(E(\bar{B}_1)\pm\bar{\delta}/1000)]$.

(d) There is a set E_3 of atoms in $\vee_0^{N-1} T^{-i}H$ whose union has measure greater than $1-\bar{\delta}/1000$ such that every h in E_3 has measure between $\exp[-N(b\pm\bar{\delta}/1000)]$ and is covered, except for a set in h of measure smaller than $\bar{\delta}m(h)/1000$ by $\exp[N(c-b\pm\bar{\delta}/500)]$ atoms of $\bar{E}_1$ with the additional property that for any of those $p\cap h$ in $\bar{E}_1$, $m(p\cap h\cap E_1\cap E_2)>\frac{1}{2}m(p\cap h)$. Call P_h the union of all the atoms of $\vee_0^{N-1} T^{-i}P$ that are thus assigned to every h in E_3.

(e) There is a set E_4 of atoms in $\vee_0^{N-1} T^{-i}$ whose union has measure greater than $1-\bar{\delta}/1000$ such that any p in E_4 has measure between $\exp[-N(a\pm\bar{\delta}/1000)]$.

(f) $N>(1000/\bar{\delta})\mathrm{Log}(200/\bar{\delta})$.

(5) Call $\bar{E}_3$ a maximal set of atoms in $\vee_0^{N-1} T^{-i}H$ which lie in E_3 such that it has been possible to assign to every h in $\bar{E}_3$ a set $\bar{P}_h$ of atoms in $\vee_0^{N-1} T^{-i}P$ in such a way as to ensure:

(i) $\bar{P}_h \subset P_h \cap E_4$ for every h in $\bar{E}_3$.

(ii) The number of elements in every $\bar{P}_h$ is greater than $\exp[N(c-b-\bar{\delta}/100)]$.

(iii) If $h_1 \neq h_2$ are two atoms in $\bar{E}_3$, then $\bar{P}_{h_1}\cap\bar{P}_{h_2}=\emptyset$.

Arguing the same way as in Lemma 2, we have that if we call E_5 the union of all $\bar{P}_h$'s for all h's in $\bar{E}_3$, $m(E_5)>1-\bar{\delta}/200$.

(6) We are now going to work separately on $h\cap\bar{P}_h$ for each fixed h in $\bar{E}_3$. Because X is a Lebesgue space, we can divide $h\cap\bar{P}_h$ into $\exp[N(u+\bar{\delta}/50)]$ sets of equal size by a partition L_h independent of the trace of $\vee_0^{N-1} T^{-i}(P\vee\bar{B}_1)$ on $h\cap\bar{P}_h$. Because of (4)(d), we have that every atom $p\cap h$ in $h\cap\bar{P}_h$ is covered for more than half its size by atoms $l\cap b$ in $L_h\cap\vee_0^{N-1} T^{-i}\bar{B}_1$ such that $p\cap h\cap b$ is in E_1, the measure of every such $l\cap b$ atom being at most $\exp[-N(c-b+\bar{\delta}/100)]$. (We have normalised the measure on $h\cap\bar{P}_h$ to be one.) The normalised measure of every $p\cap h$ atom in $h\cap\bar{P}_h$ is at least $\exp[-N(c-b+3\bar{\delta}/1000)]$, hence by (4)(f) more than twice the measure of an $l\cap b$ atom. We can now use the marriage lemma to assign to any $p\cap h$ in $h\cap\bar{P}_h$

an $l \cap b$ atom which it intersects such that $p \cap h \cap b$ is in E_1, no two different $p \cap h$ being assigned the same $l \cap b$. If we partition $h \cap \bar{P}_h$ by lumping together all the $p \cap h$ atoms that have been assigned the same b, we see that every aggregate in that partition is made of at most $\exp[N(u + \bar{\delta}/50)]$ elements. (Note that the same b can be used again in a different $h \cap \bar{P}_h$ when h moves in $\bar{E}_3$, but this will not matter, the h names being then trivially different.)

(7) If we now assign to every p in E_5 an H-N-name according to which $\bar{P}_h$ it lies in, and a $\bar{B}_1$-N-name according to which aggregate $p \cap h$ lies in $h \cap \bar{P}_h$, we can, using this assignment, if we pick (as in Lemma 2) a suitable $(P)_T$-measurable Rohlin tower of height N, produce two $(P)_T$-measurable partitions $\tilde{H}$ and $\tilde{B}$ for which

(a) $P \subset^{2\varepsilon + \delta/100} \bigvee_{-n/2}^{n/2} T^i(\tilde{H} \vee \tilde{B})$ (from (6) and (4)(a)),

(b) $d(\tilde{B}, \bar{B}_1) < \bar{\delta}_1$ (from (4)(a)),

(c) $E(\tilde{H} \vee \tilde{B}, T) > E(P, T) - u - \bar{\delta}/10$ (from (6), noticing that there are never more than $\exp[N(u + \bar{\delta}/50)]$ elements that have been assigned the same $H \vee \bar{B}_1$ name),

(d) $E(\tilde{H}, T) < E(P, T) + E(H, T) - E(P \vee H, T) + \bar{\delta}/10$ (from (5)). (b), (c) and (d) now imply that the hypotheses requested for in (3) are satisfied and we now get a $(P)_T$-measurable partition $\bar{B}$ for which the conclusions (5), (6), (7) hold, the conclusion (4) being simply (d).

Lemma 5. *Let (X, T) be an ergodic dynamical system, and H, B and Q be three finite partitions of X such that $(H)_T \perp (B)_T$, the T^iB, $i \in Z$, are independent, $(H)_T \vee (B)_T = X$ and $(Q)_T \supset (H)_T$.*

Then, for every $\varepsilon > 0$, there exists two partitions of X, $\bar{Q}$ and $\bar{B}$ such that

(1) $$|\bar{Q} - Q| < \varepsilon,$$

(2) $$(\bar{Q} \vee H, T) \sim (Q \vee H, T),$$

(3) *The $T^i\bar{B}$, $i \in Z$, are independent,*

(4) $$(\bar{B})_T \perp (\bar{Q})_T,$$

(5) $$(\bar{Q} \vee \bar{B})_T = X.$$

Proof. Let I be an abstract partition such that $E(I) = E(T) - E(Q, T)$. Then, from the relative Sinai's theorem [1], we can find a partition $\tilde{B}$ of X such that $(Q)_T \perp (\tilde{B})_T$, $d(\tilde{B}) = d(I)$ and the $T^i\tilde{B}$, $i \in Z$, are independent. Now $Q \vee \tilde{B}$ is H-conditionally finitely determined (this is true for every partition in X; see [5] prop. 4), and $E(Q \vee H \vee \tilde{B}) = E(T)$, therefore it follows from proposition 3 in [5] that there exist two partitions $\bar{Q}$ and $\bar{B}$ such that

(a) $(\bar{Q} \vee \bar{B} \vee H, T) \sim (Q \vee \tilde{B} \vee H, T)$ (whence (2)),
(b) $|\bar{Q} \vee \bar{B} - Q \vee \tilde{B}| < \varepsilon$ (whence (1)),
(c) $(\bar{Q} \vee \bar{B} \vee H)_T = X$ (whence (5) because $(Q)_T \supset (H)_T$).
Now (a) and the properties of $\tilde{B}$ imply (3) and (4).

LEMMA 6. *For every $\varepsilon > 0$ and integer $k > 1$, there exists a number $\delta > 0$ which only depends upon k and ε, for which the following property is true. Let (X, T) be an ergodic dynamical system and P and H be two finite partitions of X such that P has k elements and $E(H, T) < \delta$. Then there exists a partition $\tilde{P}$ of X with k elements such that $|\tilde{P} - P| < \varepsilon$ and $(\tilde{P})_T \supset (H)_T$.*

PROOF. We can assume, using Krieger's Theorem, that H has been taken with two elements. We then take $\delta = \varepsilon \operatorname{Log} k/20$. Let $\bar{\delta}_n$, $n \geqq 1$ be defined as $\bar{\delta}_n = \varepsilon/1000\, 2^{n+1}$. Now using McMillan's theorem and the strong Rohlin theorem, we can find a tower T_1 of height $N_1 > 2000/\varepsilon^2$ such that $m(T_1) = 1 - \delta_1 > 1 - \bar{\delta}_1$ and its base, F_1, is divided by $\bigvee_0^{N_1 - 1} T^i H$ into less than $\exp N_1 \delta$ elements. We assume $P = \{p_0, p_1, \cdots, p_{k-1}\}$ and we now define P_1 which will be a modification of P on T_1 and will have the same trace as P on T_1^c. The modification is in three steps.

(i) Because of the choice of δ we can relabel the $N_1\varepsilon/20$ first levels of T_1 such that any two different sets in $\bigvee_0^{N_1-1} T^{-i}H/F_1$ now have two different $P_1 - N_1\varepsilon/20$ names.

(ii) Put in p_0 the $N_1\varepsilon/20$ last levels in T_1.

(iii) Divide T_1 into $K < 200/\varepsilon$ towers $T_1^{(i)}$, $1 \leqq i \leqq K$, each one of them having height $N_1\varepsilon/100$ and basis $T^{(N_1\varepsilon/20 + iN_1\varepsilon/100)}F_1$, $1 \leqq i \leqq K$, the union of these towers filling T_1 except for the $N_1\varepsilon/20$ first and last levels in T_1. Partition F_1 according to the new $P - N_1$ name of points in F_1. Every set in that partition can be taken as the base of a column of height N_1 in T_1. Now for every such column, whenever its intersection with a tower $T_1^{(i)}$, $1 \leqq i \leqq K$, is completely inside p_0, put in p_1 the middle level of its intersection with $T_1^{(i)}$. The choice of N_1 forces $|P_1 - P| < \varepsilon/5$.
Now assume that $T_2, \cdots, T_n$ have been defined such that $m(T_i) = 1 - \delta_i > 1 - \bar{\delta}_i$, $2 \leqq i \leqq n$, the $P_i - N_i\varepsilon/20$ names of points in F_i (the base of T_i) determining the H-name of these points at all times t when they are in T_i but not in T_{i-1} (P_i is the modification of P which we get at stage i), every occurrence of more than $N_1\varepsilon/20$ consecutive p_0 symbols in the $P_i - N_i$ name of a point in F_i being of exactly $N_j\varepsilon/(20 \times 2^{j+1})$ consecutive p_0 symbols, the last one indicating the top level of T_j, $2 \leqq j \leqq i$, and $|P_i - P_{i-1}| < \varepsilon/5 \cdot 2^i$. The ergodic theorem and the strong Rohlin theorem now tell us that we can find a tower T_{n+1} of height $N_{n+1} > 20N_n$, with base F_{n+1} such that $m(T_{n+1}) = 1 - \delta_{n+1} > 1 - \bar{\delta}_{n+1}$, and if we partition F_{n+1}

according to the $P_n \vee \bar{T}_n - N_{n+1}$ name of its points ($\bar{T}_n$ is the partition of X into the two sets T_n, T_n^c) we get $C(n+1)$ columns of height N_{n+1} in T_{n+1}, C_{n+1}^p, $1 \leqq p \leqq C(n+1)$, every column having at most $2\delta_n N_{n+1}$ levels in T_n^c.

(a) For every p, $1 \leqq p \leqq C(n)$, we modify P_n on the first $N_{n+1}\varepsilon/20 \cdot 2^{n+1}$ levels in C_{n+1}^p in such a way that the $P_{n+1} - N_{n+1}\varepsilon/20 \cdot 2^{n+1}$ names of points in $F_{n+1} \cap C_{n+1}^p$ determine the H name of these points at the times when they are in T_n^c and in the $N_{n+1}\varepsilon/20 \cdot 2^{n+1}$ first and last levels of T_{n+1}. (This can be done because of our choice of $\bar{\delta}_n$ and the fact that $F_{n+1} \cap C_{n+1}^p$ is partitioned under the H-name at these times into less than $2^{20\delta_n N_{n+1}}$ sets, H having two elements, and the estimate in (i) holding for a proportion of more than $1-\varepsilon$ of these times.)

(b) We then declare the last $N_{n+1}\varepsilon/20 \cdot 2^{n+1}$ levels of T_{n+1} to be in p_0.

(c) The same way as in (iii), we modify P_n on every C_{n+1}^p, $1 \leqq p \leqq C(n+1)$ in such a way as to never see in any of these columns a string of length $N_1\varepsilon/40$ levels in $T_n^c \cap p_0$. This produces a new modification of P_n by less than $80\delta_n/\varepsilon N_1$. Therefore $|P_{n+1} - P_n| < \varepsilon/5 \cdot 2^{n+1}$ and $P_n \to \tilde{P}$ with $|P - \tilde{P}| < \varepsilon$. It is now easy to see that if T_i^* is the subtower of T_i on which P_i has the same trace as $\tilde{P}$, $m(T_i^*) \to 1$, and the $\tilde{P}$ name of any point in $\limsup T_i^*$ determines the H name of this point.

LEMMA 7. *Let (X, T) be an ergodic dynamical system with finite entropy, and P, H_n and B_n, $n \geqq 1$, finite partitions of X together with a sequence of positive numbers ε_n, $\varepsilon_n \downarrow 0$, be given such that:*

(1) $$(P)_T = X,$$

(2) $$(H_n)_T \perp (B_n)_T, \quad n \geqq 1,$$

(3) *The T^iB_n, $i \in Z$, are independent, $n \geqq 1$,*

(4) $$(H_n \vee B_n)_T \supset^{\varepsilon_n} P, \quad n \geqq 1,$$

(5) $$E(H_n, T) < \varepsilon_n.$$

Let Q be a finite partition of X and $\varepsilon > 0$ be given. Then there exist two finite partitions $\bar{B}$ and $\bar{Q}$ of X such that $\bar{Q}$ has the same number of elements as Q and

(5) $$|\bar{Q} - Q| < \varepsilon,$$

(6) $$(\bar{Q})_T \perp (\bar{B})_T,$$

(7) $$(\bar{Q})_T \vee (\bar{B})_T = X,$$

(8) *The $T^i\bar{B}$, $i \in Z$, are independent.*

PROOF. Call k the number of elements in Q. Let P_n be a partition $(H_n \vee B_n)_T$-measurable such that $|P_n - P| < \varepsilon_n$.

(1) If n_1 has been chosen large enough we have that

(a) Because of (1) and (4) we can find a partition $\tilde{Q}_1$ with k elements which is $(H_{n_1} \vee B_{n_1})_T$-measurable for which $|\tilde{Q}_1 - Q| < \varepsilon/8$.

(b) Because of (5) and Lemma 6 we can modify $\tilde{Q}_1$ into $\bar{Q}_1$ which still has k elements and is $(H_{n_1} \vee B_{n_1})_T$-measurable so that $|\bar{Q}_1 - \tilde{Q}_1| < \varepsilon/8$ and $(\bar{Q}_1)_T \supset (H_{n_1})_T$.

(c) Now Lemma 5 tells us that we can find two partitions $(H_{n_1} \vee B_{n_1})_T$-measurable, Q_1 and $\tilde{B}_1$ such that Q_1 has k elements, $|Q_1 - \bar{Q}_1| < \varepsilon/8$, $(Q_1)_T \perp (\tilde{B}_1)_T$, the $T^i\tilde{B}_1$, $i \in Z$, are independent and $(Q_1 \vee \tilde{B}_1)_T = (H_{n_1} \vee B_{n_1})_T$, whence $(Q_1 \vee \tilde{B}_1)_T \supset P_{n_1}$.

(d) An application of Lemma 1 now gives a number $\delta_1 > 0$ such that for any partition $\bar{Q}$ of X with k elements for which $|\bar{Q} - Q| < \delta_1$, there exists a partition $\bar{B}_1$ of X such that $(\bar{B}_1)_T \perp (\bar{Q})_T$, the $T^i\bar{B}_1$, $i \in Z$, are independent and $(\bar{Q} \vee \bar{B}_1)_T \supset^{\varepsilon_{n_1}} P_{n_1}$.

(2) We assume that we now have partitions $Q_1, Q_2, \cdots, Q_l$, $\tilde{B}_1, \tilde{B}_2 \cdots \tilde{B}_l$, integers $n_1, n_2, \cdots, n_l$, and positive numbers $\delta_1, \delta_2, \cdots, \delta_l$ such that

(a) $|Q_{j+1} - Q_j| < (\inf(\varepsilon, \delta_1, \cdots, \delta_j))/2^{j+1}$, $1 \leqq j \leqq l-1$,

(b) for every $1 \leqq j \leqq l$, $(Q_j)_T \perp (\tilde{B}_j)_T$, the $T^i\tilde{B}_j$, $i \in Z$, are independent, $(Q_j \vee \tilde{B}_j)_T \supset P_{n_j}$.

(c) for every $1 \leqq j \leqq l$, if $\bar{Q}$ is any partition with k elements such that $|\bar{Q} - Q_j| < \delta_j$ then there exists a partition $\bar{B}_j$ of X such that $(\bar{B}_j)_T \perp (\bar{Q}_T)$, the $T^i\bar{B}_j$, $i \in Z$, are independent and $(\bar{Q} \vee \bar{B}_j) \supset^{\varepsilon_{n_j}} P_{n_j}$.

(3) If we work the same way as in (1) (a) (b) (c) (d) starting with Q_l instead of Q and $(\inf(\delta_1, \cdots, \delta_l, \varepsilon))/2^{l+3}$ instead of ε, we get Q_{l+1}, $\tilde{B}_{l+1}$, n_{l+1} such that (2) (a) (b) (c) is satisfied with l changed to $l+1$. Now $Q_l \to \bar{Q}$ for which by (2) (c) and (4) we have $(\bar{Q} \vee \bar{B}_j)_T \supset^{2\varepsilon_{n_j}} P$. The conclusions (6), (7) and (8) now follow from corollary 7.1 and proposition 3 in [5], and (2) (a) implies conclusion 5.

PROPOSITION 1. *Let (X, T) be an ergodic dynamical system with finite entropy for which the weak Pinsker property is true (i.e. there exist finite partitions of X, H_n and B_n, $n \geqq 1$, such that*

$$(1) \qquad (H_{n+1})_T \subset (H_n)_T \qquad (n \geqq 1),$$

$$(2) \qquad E(H_n, T) \downarrow 0,$$

$$(3) \qquad (B_n)_T \perp (H_n)_T \qquad (n \geqq 1),$$

(4) $(B_n \vee H_n)_T = X \qquad (n \geqq 1)$,

(5) *The* $T^i B_n$, $i \in Z$, *are independent* $(n \geqq 1)$.)

Then the weak Pinsker property is true for every factor of (X, T).

PROOF. We must prove that given any finite partition Q in X, we can find two sequences of $(Q)_T$-measurable partitions $\bar{H}_n$ and $\bar{B}_n$, $n \geqq 1$, such that ($\bar{1}$) $(\bar{H}_{n+1})_T \subset (\bar{H}_n)_T$, $(n \geqq 1)$; ($\bar{2}$) $E(\bar{H}_n, T) \to 0$; ($\bar{3}$) $(\bar{B}_n)_T \perp (\bar{H}_n)_T$, $(n \geqq 1)$; ($\bar{4}$) $(\bar{B}_n \vee \bar{H}_n)_T = (Q)_T$ $(\bar{B}_n \vee \bar{H}_n)_T = (Q)_T$ $(n \geqq 1)$; ($\bar{5}$) the $T^i \bar{B}_n$, $i \in Z$, are independent, $(n \geqq 1)$. Lemma 4 and the hypothesis tell us that for any $\delta > 0$, we can find two $(Q)_T$-measurable partitions H_δ and B_δ such that (a) $(H_\delta)_T \perp (B_\delta)_T$, (b) $(H_\delta \vee B_\delta)_T \supset^\delta Q$, (c) the $T^i B_\delta$, $i \in Z$, are independent, and (d) $E(H_\delta, T) < \delta$. (First choose n_1 such that $E(H_{n_1}, T) < \delta/2$ and apply Lemma 4 to $\delta/2$, Q, H_{n_1}, and B_{n_1} to get H_δ and B_δ satisfying (a), (b) and (c) and notice that (d) comes from the conclusion (4) in Lemma 4.) Lemma 7 now ensures the existence of $\bar{H}_1$ and $\bar{B}_1$ within $(Q)_T$ such that $E(\bar{H}_1, T) < \frac{1}{2}$ and ($\bar{3}$), ($\bar{4}$) and ($\bar{5}$) hold with $n = 1$. Starting the same process in $\bar{H}_1$ yields the existence of $\bar{H}_2$ and $\tilde{B}_2$ such that $(\bar{H}_2 \vee \tilde{B}_2)_T = (\bar{H}_1)_T$, $E(\bar{H}_2)_T < \frac{1}{4}$ and ($\bar{1}$), ($\bar{3}$), ($\bar{4}$), and ($\bar{5}$) hold for $n \leqq 2$, $(\bar{B}_2 = \bar{B}_1 \vee \tilde{B}_2)$. This gives the induction procedure through which the sequences $\bar{H}_n$, $\bar{B}_n$, $n \geqq 1$, are constructed.

PROPOSITION 2. *Let* (P_n, T_n) $n \geqq 1$ *be a sequence of finite state ergodic processes that converge* $\bar{d}$ *to some process* (P, T) *and assume that for every* $n \geqq 1$ *the dynamical system generated by the* (P_n, T_n) *process satisfies the weak Pinsker property. Then the dynamical system generated by the* (P, T) *process satisfies the weak Pinsker property.*

PROOF. Because of theorem 1 in [4], there exists an ergodic dynamical system $(\bar{X}, \bar{T})$ and partitions of X, $\bar{P}_n$, $n \geqq 1$, and $\bar{P}$ such that $(\bar{P}_n, \bar{T}) \sim (P_n, T_n)$, $n \geqq 1$, $(P, T) \sim (\bar{P}, \bar{T})$ and $|\bar{P}_n - \bar{P}| \to 0$. Let $\delta > 0$ be given. Then for some n_1, $|\bar{P}_{n_1} - \bar{P}| < \delta/2$. Because the $(\bar{P}_{n_1}, \bar{T})$ process has the weak Pinsker property, we can find $\tilde{H}$ and $\tilde{B}$, $(\bar{P}_{n_1})_{\bar{T}}$-measurable partitions with the properties that $(\tilde{H})_{\bar{T}} \perp (\tilde{B})_{\bar{T}}$, $(\tilde{H} \vee \tilde{B})_{\bar{T}} = (\bar{P}_{n_1})_{\bar{T}}$, the $\bar{T}^i \tilde{B}$ are independent and with $E(\tilde{H}, \bar{T}) < \delta/2$. Then Lemma 4 gives the existence of $\bar{H}_\delta$ and $\bar{B}_\delta$ in $(\bar{P})_{\bar{T}}$ with the same properties as in the proof of Proposition 1 which now carries over the same way.

DEFINITION. Let (X, T) be an ergodic dynamical system with finite entropy and with a finite generator H for one of its factors $(H)_T$. We say that (X, T) has the weak Pinsker property relative to its factor $(H)_T$ if there exists a sequence of partitions H_n and B_n, $n \geqq 1$, such that:

(1) $(H_n)_T \supset (H)_T \qquad (n \geqq 1),$

(2) $(H_{n+1})_T \subset (H_n)_T \qquad (n \geqq 1),$

(3) $E(H_n, T \,|\, (H)_T) \downarrow 0,$

(4) $(H_n)_T \perp (B_n)_T,$

(5) The $T^i B_n$, $i \in Z$, are independent,

(6) $(H_n)_T \vee (B_n)_T = X.$

PROPOSITION 3. *Let (X, T) be an ergodic dynamical system with finite entropy and H be a finite partition of X such that (X, T) has the weak Pinsker property relative to $(H)_T$. Let Q be any partition of X such that $(Q)_T \supset (H)_T$. Then $((Q)_T, T)$ has the weak Pinsker property relative to $(H)_T$.*

PROOF. The proof comes from an imitation of the arguments in the proof of Proposition 1 which are now made "relatively to the factor $(H)_T$".

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